Brans-Dicke theory: Jordan vs Einstein Frame

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Abstract

It is well known that, in contrast to general relativity, there are two conformally related frames, the Jordan frame and the Einstein frame, in which the Brans-Dicke theory, a prototype of generic scalar-tensor theory, can be formulated. There is a long standing debate on the physical equivalence of the formulations in these two different frames. It is shown here that gravitational deflection of light to second order accuracy may observationally distinguish the two versions of the Brans-Dicke theory.

Key Words: Brans-Dicke theory, Gravitational deflection angle
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I. Introduction

The Brans-Dicke (BD) theory [1], which describes gravitation through a space-time metric ($g_{\mu\nu}$) and a massless scalar field (ϕ), is a modification or rather generalization of General Relativity (GR). The theory has recently received widespread attention due to the fact that it arises naturally as the low energy limit of many theories of quantum gravity such as the supersymmetric string theory or the Kaluza-Klein theory and is also found to be consistent with present cosmological observations [2-7].

As a generic aspect of any scalar-tensor theory, two frames are available to describe the BD theory. One frame is called the Jordan frame (JF) in which the BD field equations were originally written and the BD scalar field played the role of a spin-0 component of gravity. The other is the conformally rescaled Einstein frame (EF) in which the scalar field plays the role of a source matter field. There is a long standing debate as to whether the descriptions of the BD theory in the two frames, JF and EF, should be considered physically eqivalent. In order to get a flavor of this debate and the resulting confusion, we should only say that physicists are divided roughly into six groups depending on their attitude to the question. They can be listed as follows. Some authors: (1) neglect the issue, (2) think that the two frames are physically equivalent, (3) consider them physically nonequivalent but do not provide supporting arguments, (4) regard only JF as physical but, if necessary, use EF for mathematical convenience, (5) regard only EF as physical, (6) belong to two or more of the above categories! For a detailed account, see the review [8].

It has been argued in the literature that the physical frame is the one in which matter couples directly (as opposed to anomalously [9]) to it, particles have constant mass and move on geodesics of the physical metric so that the physical stress tensor is conserved [10]. In the non-physical frame, like the EF, particles have scalar field dependent masses and do not move along the geodesics of the EF metric due to the occurrence of a scalar field dependent force. This fact is manifest in the conservation of the sum of the energy momentum tensor in the JF, the scalar field and the cosmological term (if it is taken into account). Although, it is a matter of theoretical interpretation which frame is the "true" frame, the physical metric is still the one that defines lengths and rates of ideal clocks and it is the one that should be compared with observables.

Flanagan [11] has argued that all physical observables are conformal frame invariants. Some works in cosmology do show that it is indeed the case [12-14]. Therefore, the question arises if we can take the deflection of light as a physical observable. We state that the deflection angle is definitely a physical observable. In fact, the observed deflection of light of appropriate magnitude by solar gravity provided the first experimental proof of general relativity. The difference in the deflection angle in JF and EF is not an effect of choosing different physical units. See the end of Sec.IIIB for clarifications.

To resolve the issue in question (JF vs EF) in a more conclusive manner, we feel that it is necessary to go beyond mere (mostly speculative) theoretical arguments favoring one position or the other as listed above, and refer to a tangible *observational* ground to determine if the two frames are physically equivalent. There exist only very few works in this direction [15,16]. The situation is that the distinctive observational features that emerged from these works, such as different interaction nature of gravitational wave with gravitational detectors [15], are unlikely to be observed experimentally in the near future. Therefore, in the present work, we consider a more pragmatic premise, namely, the deflection of light by gravity up to second order in gravitational strength in both versions of the BD theory. The aim is to explore whether both formulations give same results or not.

The plan of the paper is the following: In the next section II, we briefly review the gravitational deflection of light in a generic static, spherically symmetric spacetime (in isotropic coordinates). Explicit expressions for the second order light deflection in JFBD theory and EFBD theory are obtained in section III that also includes a discussion on the matter of changing units. Finally, the results are discussed in section IV.

II. Second-order deflection angle

A general static, spherically symmetric spacetime in isotropic coordinates is given by (geometrized units are used, unless specifically restored: $G = 1$, $c = 1$)

$$ds^2 = B(\rho)dt^2 - A(\rho)\left(d\rho^2 + \rho^2 d\theta^2 + \rho^2 sin^2\theta d\varphi^2\right) \tag{1}$$

The equation of the orbital motion of test partcles can be obtained from the geodesic equations and is given by

$$\frac{d\varphi}{d\rho} = \frac{1}{\rho\sqrt{\frac{1}{Bb^2} - \frac{E}{b^2} - \frac{1}{\rho^2 A}}} \tag{2}$$

where $b = \frac{J}{E}$ is the impact parameter (the perpendicular distance between the gravitating object and the tangent to the null geodesic) at large distances, E and J are proportional to the asymptotic energy and angular momentum of the particle. Because of the spherically symmetry, the motion has been considered only in the equatorial plane ($\theta = \frac{\pi}{2}$). Following the standard treatment [17], the expression for the deflection angle for the light rays can be written as

$$\alpha(\rho_o) = I(\rho_o) - \pi \tag{3}$$

where

$$I(\rho_o) = 2\int_{\rho_o}^{\infty} \frac{d\rho}{\rho}\left[\left(\frac{\rho}{\rho_o}\right)^2 \frac{A(\rho)B(\rho_o)}{A(\rho_o)B(\rho)} - 1\right]^{-\frac{1}{2}} \tag{4}$$

ρ_o being the distance of closest approach. The relation between the impact parameter and the distance of closest approach follows from the conservation of the angular momentum of the scattering process and is given by

$$b(\rho_o) = \rho_o \sqrt{\frac{A(\rho_o)}{B(\rho_o)}} \tag{5}$$

Usually the Parameterized Post-Newtonian (PPN) formalism is employed to describe the gravitational theories in the solar system and also to compare predictions of general relativity to the results predicted by an alternative metric theory of gravity. This method actually is an approximation for obtaining the dynamics of a particle (in a weak gravitational field of a slowly moving gravitating source) to one higher order in $\frac{M}{\rho}$ than given by the Newtonian mechanics. The calculation of particle dynamics typically requires knowledge of g_{oo} more accurately than g_{ij}. But as noted in [16], understanding about the light propagation in curved spacetime to any given order needs knowledge of every term to that order.

Following the standard PPN expansion treatment, we assume that the metric tensor is equal to the Minkowski tensor $\eta_{\mu\nu}$ plus corrections in the form of expansions in powers of $\frac{M}{\rho}$ (M is the mass of the source object). Considering only up to the second-order corrections terms, we have

$$B(\rho) = 1 - 2\frac{M}{\rho} + 2\beta\frac{M^2}{\rho^2} \tag{6}$$

$$A(\rho) = 1 + 2\gamma\frac{M}{\rho} + \frac{3}{2}\delta\frac{M^2}{\rho^2} \tag{7}$$

β, γ are the PPN parameters (also known as the Eddington parameters), δ can be considered as the post-PPN parameter. Several of these parameters are different for different theories. In general relativity all of them are equal to 1 as can be readily checked by expanding the Schwarzschild metric.

The expression for the angle of light deflection up to the second order follows from Eq.(4) and is given by

$$\alpha = 2(1+\gamma)\frac{M}{\rho_o} + \left[\left(2(1+\gamma) - \beta + \frac{3}{4}\delta\right)\pi - 2(1+\gamma)^2\right]\left(\frac{M}{\rho_o}\right)^2 \tag{8}$$

It is important to note that the term representing the second order effect contains all the three parameters β, γ, and δ. So, knowing these PPN and post PPN parameters, the second order effects on deflection angle for any metric theory of gravity can be estimated readily from the above expression. For the Schwarzschild metric, the deflection angle is given by

$$\alpha = 4\frac{M}{\rho_o} + \left[\frac{15\pi}{16} - 2\right]\frac{4M^2}{\rho_o^2} \tag{9}$$

A limitation of the expression (8) is that it depends on the coordinate variable ρ. However, it can also be expressed in terms of coordinate independent variables, such as the impact parameter. In that case, the deflection angle reduces to

$$\alpha = 2(1+\gamma)\frac{M}{b} + \left[2(1+\gamma) - \beta + \frac{3}{4}\delta\right]\frac{\pi M^2}{b^2} \tag{10}$$

III. Deflection angle in the BD theory

The expressions of the Eddington parameters β and γ for the theories under investigation are already known. For the BD theory in the Jordan frame (JFBD) these two PPN parameters are $\beta = 1$, $\gamma = \frac{\omega+1}{\omega+2}$, whereas for the BD theory in the Einstein frame (EFBD) both parameters are equal to 1. So our main task is to calculate the post PPN parameter δ for these theories. The parameter δ occurs only in the metric coefficient g_{ij}. So it is enough for us to consider only the static case.

A. The JFBD theory

The scalar field in JFBD theory acts as the source of the (local) gravitational coupling with $G \sim \phi^{-1}$. As a consequence, the gravitational "constant" is not in fact a constant but is determined by the total matter in the universe through an auxiliary scalar field equation. The scalar field couples to both matter and spacetime geometry and the strength of the coupling is represented by a single dimensionless constant parameter ω. It is generally considered that under the limit $\omega \rightarrow \infty$, the vacuum (or for traceless matter field) BD theory (and its dynamic generalization) reduces to the GR but the recent finding suggests that such a convergence is not always true [18-22].
In the Jordan conformal frame, the BD action takes the form

$$\mathcal{A} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(\phi R - \frac{\omega}{\phi} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + \mathcal{L}_{matter} \right) \tag{11}$$

where $\mathcal{L}_{matter}$ is the Lagrangian density of ordinary matter and R is the Ricci scalar. As stated earlier, the theory is constrained by the solar system experiments. The recent conjunction experiment with Cassini spacecraft constrains the value of the coupling constant as $|\omega| > 5 \times 10^4$ [23].
The static spherically symmetric matter free solution of the BD theory in isotropic coordinates is given by [24,25]:

$$ds^2 = + \left(\frac{1 - \frac{B}{\rho}}{1 + \frac{B}{\rho}} \right)^{\frac{2}{\lambda}} dt^2 - \left(1 + \frac{B}{\rho} \right)^4 \left(\frac{1 - \frac{B}{\rho}}{1 + \frac{B}{\rho}} \right)^{\frac{2(\lambda - C - 1)}{\lambda}} \left(d\rho^2 + \rho^2 d\theta^2 + \rho^2 sin^2\theta d\phi^2 \right) \tag{12}$$

$$\phi = \phi_0 \left(\frac{1 - \frac{B}{\rho}}{1 + \frac{B}{\rho}} \right)^{\frac{C}{\lambda}} \tag{13}$$

with

$$\lambda^2 = (C+1)^2 - C \left(1 - \frac{\omega C}{2} \right) \tag{14}$$

where B, C are constants of integration. By the weak field Newtonian approximation, we can set $\frac{4B}{\lambda} = 2GM$, where G is the gravitational constant measured by a Cavendish or a similar experiment and M is the gravitating mass. Further, by matching the interior and exterior (due to physically reasonable spherically symmetric matter source) scalar fields, the constant C can be identified as $C = \frac{1}{\omega+2}$[17]. These are the standard procedues for fixing constants of a metric theory of gravity. Expanding the metric coefficients and retaining only up to the second order terms in $\frac{M}{\rho}$, we get the parameter δ as

$$\delta = 1 - \frac{15\omega + 22}{6(\omega+2)^2} \tag{15}$$

Hence, finally, the deflection angle becomes

$$\alpha = \left(\frac{2\omega+3}{2\omega+4}\right)\frac{4M}{\rho_o} + \left[\left(\frac{2\omega+3}{2\omega+4} - \frac{15\omega+22}{8(2\omega+4)^2} - \frac{1}{16}\right)\pi - 2\left(\frac{2\omega+3}{2\omega+4}\right)^2\right]\frac{4M^2}{\rho_o^2} \tag{16}$$

In the limit $\omega \to \infty$, the above expression reduces to the general relativity value. The deflection angle can also be readily expressed in terms of impact parameter using Eqs. (10) and (15).

B. The EFBD theory

Recent cosmological observations indicate that the universe is undergoing cosmic acceleration and is dominated by a dark energy component with negative pressure [26-29]. Cosmological constant (Λ) is a straightforward and natural candidate for such a component. However, the observational upper limit on Λ is more than 120 orders smaller than what is expected naturally from a vacuum energy originating at the Planck time. An alternative realization of dark energy is in the form of a minimally coupled scalar field ϕ with a specific potential $U(\phi)$ (the so called 'quintessence') whose slowly varying energy density would mimic an effective cosmological constant. This is very reminiscent of the mechanism producing the inflationary phase. A minimally coupled scalar field is, thus, an attractive possibility in modern cosmology.
The action for the EFBD theory is

$$\mathcal{A} = \int \sqrt{-\tilde{g}} d^4x \left(\tilde{R} + \mu \tilde{g}^{\alpha\beta} \tilde{\phi}_{,\alpha} \tilde{\phi}_{,\beta}\right) \tag{17}$$

This action is obtained from the action (11) by conformal transformation of the metric $\tilde{g}_{\alpha\beta} = \phi g_{\alpha\beta}$ and a redefinition of the scalar $\widetilde{\phi} = \left(\frac{2\omega+3}{2\mu}\right)^{1/2} \ln\phi$. The extra constant μ is introduced here to fix the sign of the kinetic term, but it does not appear in metric observations.

A static spherically symmetric vacuum solution to the EFBD theory (with the cosmological constant $\Lambda = 0$) is the well known Buchdahl solution [30] which is also variously referred (as demonstrated in [31]) to as JNW [32] or Wyman solution [33]. Like its counterpart (Schwarzschild solution) in GR, this solution also correctly explains all the post-Newtonian tests of GR. However, in contrast

to the Schwarzschild solution, Buchdahl solution does not represent a black hole spacetime but possesses a strong globally naked singularity, respecting the "scalar no hair theorem" [34] which purports to exclude the availability of any knowledge of a scalar field from the exterior of a spherically symmetric black hole. Whether a naked singularity occurs generically in a physically realistic collapse is a subject of considerable debate [35].
The Buchdahl solution [30], in isotropic coordinates, is given by

$$ds^2 = \left(\frac{1 - \frac{m}{2\rho}}{1 + \frac{m}{2\rho}} \right)^{2\xi} dt^2 - \left(1 - \frac{m}{2\rho}\right)^{2(1-\xi)} \left(1 + \frac{m}{2\rho}\right)^{2(1+\xi)} [d\rho^2 + \rho^2(d\theta^2 + sin^2\theta d\phi^2)], \tag{18}$$

and the expression for the scalar field is given by

$$\phi(\rho) = \sqrt{\frac{2(1-\xi^2)}{\mu}} ln \left(\frac{1 - \frac{m}{2\rho}}{1 + \frac{m}{2\rho}} \right), \ \rho > m/2 \tag{19}$$

The Arnowitt-Deser-Misner (ADM) mass of the source corresponding to the above solution is given by $M = \xi m$. The effect of scalar field is usually described in terms of a scalar charge defined as $q = m\sqrt{\frac{2(1-\xi^2)}{\mu}}$. Expanding the metric coefficients in $\frac{M}{\rho}$ and comparing with Eqs.(6) and (7), we get

$$\delta = 1 - \frac{1}{3}(1 - \xi^2) \tag{20}$$

Thus the final expression for the second-order deflection angle becomes

$$\alpha = 4\frac{M}{\rho_o} + \left[\frac{1}{16}(14 + \xi^2)\pi - 2 \right] \frac{4M^2}{\rho_o^2} \tag{21}$$

One can obtain the general relativity result by taking $\xi = 1$.

Usually, a conformal transformation is regarded as a change in physical units. Hence, a natural question is whether the difference between the deflection angles in JF and EF, as revealed from Eqs.(16) and (21) respectively, is an effect of selecting different conventions of physical units. To see that this is not the case here, we get from (12), with $\frac{4B}{\lambda} = 2GM$, to first order, $g_{tt}^{JF} = 1 - \frac{2GM}{\rho}$ and $g_{\rho\rho}^{JF} = 1 + \frac{2(C+2)GM}{\rho}$. Now, via the conformal transformation $g_{\alpha\beta}^{EF} = \phi g_{\alpha\beta}^{JF}$, we get, to first order, $g_{tt}^{EF} = 1 - \frac{2B(C+2)}{\lambda\rho}$ and $g_{\rho\rho}^{EF} = 1 + \frac{2B(C+2)}{\lambda\rho}$. If we think that EF is the physical frame, then, again by standard Newtonian identification, $\frac{2B(C+2)}{\lambda} = 2GM$, we get $g_{tt}^{EF} = 1 - \frac{2GM}{\rho}$, $g_{\rho\rho}^{EF} = 1 + \frac{2GM}{\rho}$. [Using the relation $\xi = \frac{1}{\lambda}\left(\frac{C+2}{2}\right)$ and $B = \frac{mG}{2}$, we do get $M = \xi m$]. Clearly, just by changing units, the components of the metric tensors in JF and EF can not be reconciled even in the first order. One of the underlying reasons could be that the numerical values of scalar invariants, like the Ricci scalar, change under conformal transformations. Another reason could be that the conformal transformation from JF to EF and its inverse do not preserve the exact specific form of either action.

IV. Discussion

Our main observations are as follows:
(1) The JFBD theory contains an adjustable coupling parameter ω. As ω increases, the post-Newtonian expansions of the BD theory increasingly approach the corresponding GR expressions. As a result, observations can not rule out the JFBD theory in favor of GR, but can only place limits on the coupling parameter ω. Using the present lower bound on ω as obtained from the recent conjunction experiment with Cassini spacecraft, we found that the second order deflection angles of light in the GR and in the JFBD theory are *same* up to an accuracy of 100 *pico arc seconds*. Hence the proposed experiments for measuring the deflection of light to second order accuracy, such as the LATOR experiment [36,37] which is expected to achieve an accuracy of nearly 10 *nano arc second* in angular measurement, would not impose any further constraint on the coupling parameter ω. However, it should be noted that, from the accurate observation of the first order deflection of light, the LATOR mission will measure the PPN parameter γ very precisely which in turn will provide better information on the value of ω. Similar conclusions will hold also for the generalized scalar-tensor theories for which $\omega = \omega(\phi)$.
(2) The EFBD theory contains a scalar field that couples minimally to gravity. Then, at the first PPN order, there is no effect of the scalar field in the deflection angle. The difference from GR occurs only in the magnitudes of the second and higher orders. We observe that the second order deflection angle depends on scalar charge in addition to the ADM mass of the source object and the bending is reduced under the effect of the scalar charge. If the scalar charge is just 10% of the total ADM mass of the sun, the *difference* between the deflection angle (up to second order) of light in the Schwarzschild and in the Buchdahl spacetime is around 7 *nano arc sec*. So the LATOR mission, for the first time, might detect signatures of the minimally coupled scalar field. The difference is significant compared to the much lesser (in principle zero, as ω is increased without limit) difference between the Schwarzschild and JFBD theory and its measurement could observationally distinguish between JF and EF. This is what we wanted to argue in this paper.
3) We have not touched upon the cosmological issues. It might be interesting to know if tests at the solar system level be used to set the the boundary conditions for a cosmological problem.

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